

# ON THE SEMI-REGULAR MODULE AND VERTEX OPERATOR ALGEBRAS

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## 1. INTRODUCTION

The aim of this paper is to give a proof of a conjecture stated in a previous paper by the author ([Z1]).

Let  $\mathfrak{g}$  be a simple complex Lie algebra,  $\hat{\mathfrak{g}}$  be the affine Lie algebra and  $h^\vee$  be the dual Coxeter number of  $\mathfrak{g}$ . Let  $\mathcal{A}_{\mathfrak{g},k}$  be the vertex algebroid associated to  $\mathfrak{g}$  and a complex number  $k$ , according to [GMS1], we can construct a vertex algebra  $U\mathcal{A}_{\mathfrak{g},k}$ , called the enveloping algebra of  $\mathcal{A}_{\mathfrak{g},k}$ . Set  $\mathbb{V} = U\mathcal{A}_{\mathfrak{g},k}$ . It is shown in [AG] and [GMS2] that not only  $\mathbb{V}$  is a  $\hat{\mathfrak{g}}$ -representation of level  $k$ , it is also a  $\hat{\mathfrak{g}}$ -representation of the dual level  $\bar{k} = -2h^\vee - k$ . Moreover the two copies of  $\hat{\mathfrak{g}}$ -actions commute with each other, i.e.  $\mathbb{V}$  is a  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -representation.

When  $k \notin \mathbb{Q}$ , the vertex operator algebra  $\mathbb{V}$  decomposes into

$$\bigoplus_{\lambda \in P^+} V_{\lambda,k} \otimes V_{\lambda^*,\bar{k}}$$

as a  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -module (see [FS], [Z1]). Here  $P^+$  is the set of dominant integral weights of  $\mathfrak{g}$ ,  $V_{\lambda,k}$  is the Weyl module induced from  $V_\lambda$ , the irreducible representation of  $\mathfrak{g}$  with highest weight  $\lambda$ , in level  $k$ , and  $V_{\lambda^*,\bar{k}}$  is induced from  $V_{\lambda^*}$  in the dual level  $\bar{k}$ . In fact the vertex operators can be constructed using intertwining operators and Knizhnik-Zamolodchikov equations (see [Z1]).

In the case where  $k \in \mathbb{Q}$ , the  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -module structure of  $\mathbb{V}$  is much more complicated. In the present paper, we prove a result about the existence of canonical filtrations of  $\mathbb{V}$  conjectured at the end of [Z1]. More precisely we will prove the following.

**Theorem 1.** *Let  $k \in \mathbb{Q}$ ,  $k > -h^\vee$ . The vertex operator algebra  $\mathbb{V}$  admits an increasing (resp. a decreasing) filtration of  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -submodules with factors isomorphic to*

$$V_{\lambda,k} \otimes V_{\lambda,\bar{k}}^c \quad (\text{resp. } V_{\lambda,k}^c \otimes V_{\lambda,\bar{k}}), \quad \lambda \in P^+,$$

where  $V_{\lambda,\bar{k}}^c$  is the contragredient module of  $V_{\lambda,\bar{k}}$  defined by the anti-involution:  $x(n) \mapsto -x(-n)$ ,  $\underline{c} \mapsto \underline{c}$  of  $\hat{\mathfrak{g}}$ .

We need two ingredients to prove the theorem: one is the semi-regular module; the other is the regular representation of the corresponding quantum group at a root of unity.

The standard semi-regular module was first introduced by A. Voronov in [V] to treat the semi-infinite cohomology of infinite dimensional Lie algebras as a two-sided derived functor of a functor that is neither left nor right exact. It was also studied rigorously by S. M. Arkhipov. He defined the associative algebra semi-infinite cohomology in

the derived categories' setting (see [A1]), and discovered a deep semi-infinite duality which generalizes the classical bar duality of graded associative algebras (see [A2]).

The semi-regular module  $S_\gamma$  associated to a semi-infinite structure  $\gamma$  of  $\hat{\mathfrak{g}}$  (see [V]) is the semi-infinite analogue of the universal enveloping algebra  $U$  of  $\hat{\mathfrak{g}}$ . In particular  $S_\gamma$  is a  $U$ -bimodule, and the tensor product  $S_\gamma \otimes_U \mathbb{V}$  becomes a  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -representation. We will show in Section 3 that  $S_\gamma \otimes_U \mathbb{V}$  can be embedded into  $U^*$  as a bisubmodule. In fact it is spanned by the matrix coefficients of modules from the category  $\mathcal{O}_{\bar{k}+h^\vee}$ , defined and studied by Kazhdan and Lusztig in [KL1-4] for  $\bar{k} < -h^\vee$ .

In the series of papers [KL1-4], Kazhdan and Lusztig defined a structure of braided category on  $\mathcal{O}_{\bar{k}+h^\vee}$ , and constructed an equivalence between the tensor category  $\mathcal{O}_{\bar{k}+h^\vee}$  and the category of finite dimensional integrable representations of the quantum group with parameter  $e^{i\pi/(\bar{k}+h^\vee)}$  (in the simply-laced case). It motivated the author to study the structure of regular representations of the quantum group at roots of unity (see [Z2]).

One of the main results in [Z2] is that the quantum function algebra admits an increasing filtration of (bi)submodules such that the subquotients are isomorphic to the tensor products of the dual of Weyl modules  $W_{-\omega_0\lambda}^* \otimes W_\lambda^*$  ( $\omega_0$  being the longest element in the Weyl group). Translating this to the affine Lie algebra, it means that  $S_\gamma \otimes_U \mathbb{V}$  admits an increasing filtration of  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -submodules with factors isomorphic to  $V_{-\omega_0\lambda, \bar{k}}^* \otimes V_{\lambda, \bar{k}}^c$ . Applying the functor  $\mathcal{H}om_U(S_\gamma, -)$  (see [S, Theorem 2.1]) to this filtration of  $S_\gamma \otimes_U \mathbb{V}$ , we obtain an increasing filtration of  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -submodules of the vertex operator algebra  $\mathbb{V}$  with factors described in Theorem 1. The corresponding decreasing filtration is obtained by using the non-degenerate bilinear form on  $\mathbb{V}$  constructed in [Z1].

The paper is organized as follows: In Section 2, we follow [S] to recall the definition of semi-regular module  $S_\gamma$  and the two functors defined with it. In Section 3, we embed  $S_\gamma \otimes_U \mathbb{V}$  into the dual of  $U$  as a (bi)submodule. In Section 4, we prove the main theorem about the filtrations of the vertex operator algebra  $\mathbb{V}$  using results of [Z2].

## 2. SEMI-REGULAR MODULE $S_\gamma$ AND EQUIVALENCE OF CATEGORIES

The semi-regular module of a graded Lie algebra with a semi-infinite structure was first introduced by A. Voronov in [V], where it was called the “standard semijective module”. It replaces the universal enveloping algebra (and its dual) in the semi-infinite theory, and like the universal enveloping algebra, it possesses left and right (semi)regular representations. Voronov used semijective complexes and resolutions to define the semi-infinite cohomology of infinite dimensional Lie algebras as a two-sided derived functor of a functor that is intermediate between the functors of invariants and coinvariants.

In [A2], S. M. Arkhipov generalized the classical bar duality of graded associative algebras to give an alternative construction of the semi-infinite cohomology of associative algebras. Given a graded associative algebra  $A$  with a triangular decomposition, he introduced the endomorphism algebra  $A^\sharp$  of a semi-regular  $A$ -module  $S_A$  (see [A1]). In the case where  $A$  is the universal enveloping algebra of a graded Lie algebra, the algebra  $A^\sharp$  is also a universal enveloping algebra of a Lie algebra which differs from the previous one by a 1-dimensional central extension (determined by

the critical 2-cocycle). In the affine Lie algebra case, he proved that the category of all  $\hat{\mathfrak{g}}$ -modules with a Weyl filtration in level  $k$  is contravariantly equivalent to the analogous category in the dual level  $\bar{k}$ . This equivalence was obtained directly in [S], where W. Soergel used it to find characters of tilting modules of affine Lie algebras and quantum groups.

Let us recall the definition of the semi-regular module from [S, Theorem 1.3].

Let  $\mathfrak{g}$  be a simple complex Lie algebra. Let  $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\underline{c}$  be the affine Lie algebra, where the commutator relations are given by

$$[x(m), y(n)] = [x, y](m+n) + m\delta_{m+n,0}(x, y)\underline{c}.$$

Here  $x(n) = x \otimes t^n$  for  $x \in \mathfrak{g}$ ,  $(,)$  is the normalized Killing form on  $\mathfrak{g}$  and  $\underline{c}$  is the center. Define a  $\mathbb{Z}$ -grading on  $\hat{\mathfrak{g}}$  by  $\deg x(n) = n$  and  $\deg \underline{c} = 0$ .

Set  $\hat{\mathfrak{g}}_{>0} = \mathfrak{g} \otimes t\mathbb{C}[t]$ ,  $\hat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ ,  $\hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}\underline{c}$  and  $\hat{\mathfrak{g}}_{\geq 0} = \hat{\mathfrak{g}}_{>0} \oplus \hat{\mathfrak{g}}_0$ . Denote the enveloping algebras of  $\hat{\mathfrak{g}}, \hat{\mathfrak{g}}_{\geq 0}, \hat{\mathfrak{g}}_{<0}$  by  $U, B, N$ . Obviously  $U, B, N$  inherit  $\mathbb{Z}$ -gradings from the corresponding Lie algebras.

Define a character

$$\gamma : \hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}\underline{c} \rightarrow \mathbb{C}; \quad \gamma|_{\mathfrak{g}} = 0, \quad \gamma(\underline{c}) = 2h^\vee,$$

where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}$ . It is easy to check that  $\gamma$  is a semi-infinite character for  $\hat{\mathfrak{g}}$  (see [S, Definition 1.1]).

For any two  $\mathbb{Z}$ -graded vector spaces  $M, M'$ , define the  $\mathbb{Z}$ -graded vector space  $\mathcal{H}om_{\mathbb{C}}(M, M')$  with homogeneous components

$$\mathcal{H}om_{\mathbb{C}}(M, M')_j = \{f \in \text{Hom}_{\mathbb{C}}(M, M') | f(M_i) \subset M'_{i+j}\}.$$

The graded dual  $N^{\otimes} = \oplus_i N_i^*$  of  $N$  is an  $N$ -bimodule via the prescriptions  $(nf)(n_1) = f(n_1n)$  and  $(fn)(n_1) = f(nn_1)$  for any  $n, n_1 \in N$ ,  $f \in N^{\otimes}$ . We have  $N^{\otimes} = \mathcal{H}om_{\mathbb{C}}(N, \mathbb{C})$ , if we equip  $\mathbb{C}$  with the  $\mathbb{Z}$ -grading  $\mathbb{C} = \mathbb{C}_0$ .

Consider the following sequence of isomorphisms of ( $\mathbb{Z}$ -graded) vector spaces:

$$\mathcal{H}om_B(U, \mathbb{C}_\gamma \otimes_{\mathbb{C}} B) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{C}}(N, B) \xleftarrow{\sim} N^{\otimes} \otimes_{\mathbb{C}} B \xrightarrow{\sim} N^{\otimes} \otimes_N U,$$

here  $\mathbb{C}_\gamma$  is the one-dimensional representation of  $\hat{\mathfrak{g}}_{\geq 0}$  defined by the character  $\gamma : \hat{\mathfrak{g}}_0 \rightarrow \mathbb{C}$  and the surjection  $\hat{\mathfrak{g}}_{\geq 0} \twoheadrightarrow \hat{\mathfrak{g}}_0$ , and  $\mathbb{C}_\gamma \otimes_{\mathbb{C}} B$  is the tensor product of these two representations as a left  $\hat{\mathfrak{g}}_{\geq 0}$ -module. In the leftmost term,  $U$  is considered a  $B$ -module via left multiplication of  $B$  on  $U$ , and  $\mathcal{H}om_B(U, \mathbb{C}_\gamma \otimes_{\mathbb{C}} B)$  is made into a (left)  $U$ -module via the right multiplication of  $U$  onto itself. The first isomorphism is defined as the restriction to  $N$  using the identification  $\mathbb{C}_\gamma \otimes_{\mathbb{C}} B \xrightarrow{\sim} B; 1 \otimes b \mapsto b$ .

As a vector space, the semi-regular module

$$S_\gamma = N^{\otimes} \otimes_{\mathbb{C}} B.$$

It is also a  $U$ -bimodule: the left (resp. right)  $U$ -action on  $S_\gamma$  is defined via the first two (resp. last) isomorphisms. The semi-infinite character  $\gamma$  ensures that these two actions commute.

**Lemma 2.1.**  $\underline{c} \cdot s = s \cdot \underline{c} + 2h^\vee s$  for any  $s \in S_\gamma$ , where  $\underline{c} \cdot s$  and  $s \cdot \underline{c}$  stand for the left and right actions of  $\underline{c}$  on  $s \in S_\gamma$ .

*Proof.* Easily verified. □

**Proposition 2.2.** *[S, Theorem 1.3] The map  $\iota : N^\otimes \hookrightarrow S_\gamma; f \mapsto f \otimes 1$  is an inclusion of  $N$ -bimodules. The maps  $U \otimes_N N^\otimes \rightarrow S_\gamma; u \otimes f \mapsto u \cdot \iota(f)$  and  $N^\otimes \otimes_N U \rightarrow S_\gamma; f \otimes u \mapsto \iota(f) \cdot u$  are bijections.*

**Remark 2.3.** The sequence of isomorphisms

$$S_\gamma = U \otimes_N N^\otimes \cong B \otimes_{\mathbb{C}} N^\otimes \cong \text{Hom}_{\mathbb{C}}(N, B) \xrightarrow{\sim} \text{Hom}_{B\text{-right}}(U, \mathbb{C}_{-\gamma} \otimes B)$$

induces a right  $U$ -map from  $S_\gamma$  to  $\text{Hom}_{B\text{-right}}(U, \mathbb{C}_{-\gamma} \otimes B)$ . The right  $U$ -module structure of the latter is given by the left multiplication of  $U$  on the first argument in  $\text{Hom}$ .

Let  $P^+$  be the dominant integral weights of  $\mathfrak{g}$  and  $\lambda \in P^+$ . Denote by  $V_{\lambda,k} = \text{Ind}_{\hat{\mathfrak{g}}_{\geq 0}}^{\hat{\mathfrak{g}}} V_\lambda$  the Weyl module induced from the finite dimensional irreducible representation of  $\mathfrak{g}$  with highest weight  $\lambda$  in level  $k$ . Let  $V_{\lambda,k}^*$  be the graded dual of  $V_{\lambda,k}$ , on which  $\hat{\mathfrak{g}}$  acts by  $Xf(v) = -f(Xv)$  for any  $X \in \hat{\mathfrak{g}}$ ,  $f \in V_{\lambda,k}^*$  and  $v \in V_{\lambda,k}$ .

Let  $\mathcal{M}$  (resp.  $\mathcal{K}$ ) denote the category of all  $\mathbb{Z}$ -graded representations of  $\hat{\mathfrak{g}}$ , which are over  $N$  isomorphic to finite direct sums of may-be grading shifted copies of  $N$  (resp.  $N^\otimes$ ). In fact  $\mathcal{M}$  (resp.  $\mathcal{K}$ ) consists precisely of those  $\mathbb{Z}$ -graded  $\hat{\mathfrak{g}}$ -modules, which admit a finite filtration with factors isomorphic to Weyl modules (resp. the dual of Weyl modules) (see [S, Remarks 2.4]).

**Proposition 2.4.** *[S, Theorem 2.1] The functor  $S_\gamma \otimes_U - : \mathcal{M} \rightarrow \mathcal{K}$  defines an equivalence of categories with inverse  $\text{Hom}_U(S_\gamma, -)$ , such that short exact sequences correspond to short exact sequences.*

*Proof.* Note that  $S_\gamma \otimes_U - \cong N^\otimes \otimes_N -$  and  $\text{Hom}_U(S_\gamma, -) \cong \text{Hom}_N(N^\otimes, -)$  by Proposition 2.2.  $\square$

**Proposition 2.5.** *Let  $E$  be a  $\mathbb{Z}$ -graded  $B$ -module bounded from below, the functor  $S_\gamma \otimes_U -$  maps  $U \otimes_B E$  to  $\text{Hom}_B(U, \mathbb{C}_\gamma \otimes E)$ .*

*Proof.* Similar to the construction of the semi-regular module  $S_\gamma$ , consider the following sequence of isomorphisms of  $\mathbb{Z}$ -graded vector spaces:

$$S_\gamma \otimes_U (U \otimes_B E) \cong N^\otimes \otimes_{\mathbb{C}} E \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(N, E) \xleftarrow{\sim} \text{Hom}_B(U, \mathbb{C}_\gamma \otimes E).$$

It is straightforward to check that, under these isomorphisms, the (left)  $U$ -module structure of  $S_\gamma \otimes_U (U \otimes_B E)$  agrees with that of  $\text{Hom}_B(U, \mathbb{C}_\gamma \otimes E)$ .  $\square$

**Remark 2.6.** In general for any  $\mathbb{Z}$ -graded  $B$ -module  $E'$ , the inclusion  $S_\gamma \otimes_U (U \otimes_B E') \cong N^\otimes \otimes_{\mathbb{C}} E' \hookrightarrow \text{Hom}_B(U, \mathbb{C}_\gamma \otimes E')$  is a  $U$ -map.

**Proposition 2.7.** *Let  $F$  be a  $\mathbb{Z}$ -graded  $B$ -module bounded from above, then the functor  $\text{Hom}_U(S_\gamma, -)$  maps  $\text{Hom}_B(U, F)$  to  $U \otimes_B (\mathbb{C}_{-\gamma} \otimes F)$ .*

*Proof.* The isomorphism of vector spaces  $U \otimes_B (\mathbb{C}_{-\gamma} \otimes F) \xrightarrow{\sim} \text{Hom}_U(S_\gamma, \text{Hom}_B(U, F))$ , induced from

$$\begin{aligned} \text{Hom}_U(S_\gamma, \text{Hom}_B(U, F)) &\cong \text{Hom}_N(N^\otimes, \text{Hom}_{\mathbb{C}}(N, F)) \\ &\cong \text{Hom}_{\mathbb{C}}(N^\otimes, F) \cong N \otimes_{\mathbb{C}} F \cong U \otimes_B (\mathbb{C}_{-\gamma} \otimes F), \end{aligned}$$

agrees with the composition of (left)  $U$ -maps

$$U \otimes_B (\mathbb{C}_{-\gamma} \otimes F) \rightarrow \text{Hom}_U(S_\gamma, S_\gamma \otimes_U (U \otimes_B (\mathbb{C}_{-\gamma} \otimes F))) \rightarrow \text{Hom}_U(S_\gamma, \text{Hom}_B(U, F)),$$

hence it is a  $U$ -isomorphism.  $\square$

In particular  $S_\gamma \otimes_U -$  transforms Weyl modules to the dual of Weyl modules, and  $\mathcal{H}om_U(S_\gamma, -)$  transforms the latter to the former (both with a level shift).

**Corollary 2.8.**  $S_\gamma \otimes_U V_{\lambda,k} \cong V_{\lambda^*,\bar{k}}^*$ , and  $\mathcal{H}om_U(S_\gamma, V_{\lambda,\bar{k}}^*) \cong V_{\lambda^*,k}$ , here  $\lambda^*$  denotes the highest weight of  $V_\lambda^*$ .

*Proof.* Note that  $U \otimes_B V_\lambda = V_{\lambda,k}$  and  $\mathcal{H}om_B(U, \mathbb{C}_\gamma \otimes V_\lambda) \cong V_{\lambda^*,\bar{k}}^*$  if  $\underline{c}$  acts on  $V_\lambda$  as scalar multiplication by  $k$ .  $\square$

### 3. REALIZATION OF $S_\gamma \otimes_U \mathbb{V}$ INSIDE $U^*$

Fix a complex number  $k$ , and let  $\mathbb{V} = U\mathcal{A}_{\mathfrak{g},k}$  be the vertex operator algebra associated to the vertex algebroid  $\mathcal{A}_{\mathfrak{g},k}$  (see [AG], [GMS1, 2], [Z1]). Note that in [Z1], we used  $\mathbb{V}$  to denote the vertex operator algebra for generic values of  $k \notin \mathbb{Q}$ , but here we adopt this notation with no restriction on  $k$ .

The vertex operator algebra  $\mathbb{V}$  admits two commuting actions of  $\hat{\mathfrak{g}}$  in dual levels  $k, \bar{k} = -2h^\vee - k$ . It follows from Lemma 2.1 that  $S_\gamma \otimes_U \mathbb{V}$ , using the  $\hat{\mathfrak{g}}_k$ -module structure of  $\mathbb{V}$ , becomes a  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -representation. Define  $U(\hat{\mathfrak{g}}, k) = U(\hat{\mathfrak{g}})/(\underline{c} - k)U(\hat{\mathfrak{g}})$ . Our goal is to construct an embedding of  $U$ -bimodules

$$\Phi : S_\gamma \otimes_U \mathbb{V} \hookrightarrow U(\hat{\mathfrak{g}}, k)^*.$$

Let  $\mathbb{B} = \oplus_{i \leq 0} \mathbb{B}_i$  (denoted by “ $B$ ” with opposite grading in [Z1]) be the commutative vertex subalgebra of  $\mathbb{V}$  generated by  $A$ , where  $A$  is the commutative algebra of regular functions on an affine connected algebraic group  $G$  with Lie algebra  $\mathfrak{g}$ . Recall that  $\mathbb{B}$  is closed under the actions of  $U(\hat{\mathfrak{g}}_{\geq 0}, k)$  and  $U(\hat{\mathfrak{g}}_{\geq 0}, \bar{k})$ . As a  $\hat{\mathfrak{g}}_k$ -module, we have  $\mathbb{V} \cong U \otimes_B \mathbb{B} \cong N \otimes_{\mathbb{C}} \mathbb{B}$  (see e.g. [Z1, Proposition 3.16]). Since  $S_\gamma \cong N^{\oplus} \otimes_N U$  as a right  $U$ -module, we have

$$S_\gamma \otimes_U \mathbb{V} \cong N^{\oplus} \otimes_N \mathbb{V} \cong N^{\oplus} \otimes_{\mathbb{C}} \mathbb{B}.$$

Define a functional  $\epsilon : \mathbb{B} \rightarrow \mathbb{C}$  as follows:  $\epsilon|_{\mathbb{B}_{\leq -1}} = 0$  and its restriction to  $\mathbb{B}_0 = A$  is the evaluation of functions at identity.

Multiplication induces isomorphism of vector spaces:  $N \otimes_{\mathbb{C}} B \cong U$ , hence any  $u \in U$  can be written as  $u = u_{<0} u_{\geq 0}$  with  $u_{<0} \in N$  and  $u_{\geq 0} \in B$ .

Let  $\bar{\cdot} : U \rightarrow U; u \rightarrow \bar{u}$  be the anti-involution of  $U$  determined by  $-\text{Id} : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ . Define a map

$$\Phi : S \otimes_U \mathbb{V} \rightarrow U^*$$

as follows: for any  $f \in N^{\oplus}$ ,  $b \in \mathbb{B}$ ,

$$\Phi(f \otimes b)(u_{<0} u_{\geq 0}) = f(\overline{u_{<0}}) \epsilon(u_{\geq 0}^r \cdot b),$$

here  $u_{\geq 0}^r \cdot b$  means the  $U(\hat{\mathfrak{g}}_{\geq 0}, \bar{k})$ -action on  $\mathbb{B}$ . In fact  $\Phi(f \otimes b) \in U(\hat{\mathfrak{g}}, k)^*$ .

The dual space  $U^*$  is a  $U$ -bimodule via the recipes  $(u \cdot g)(u_1) = g(u_1 u)$  and  $(g \cdot u)(u_1) = g(u u_1)$  for any  $u, u_1 \in U$ ,  $g \in U^*$ .

**Theorem 3.1.** *For any  $u \in U$  and  $f \otimes b \in N^{\oplus} \otimes_{\mathbb{C}} \mathbb{B}$  ( $\cong S_\gamma \otimes_U \mathbb{V}$ ), we have*

$$\Phi(u^l \cdot (f \otimes b)) = (\Phi(f \otimes b)) \cdot \bar{u},$$

$$\Phi(u^r \cdot (f \otimes b)) = u \cdot (\Phi(f \otimes b)),$$

here  $u^l \cdot (f \otimes b)$ ,  $u^r \cdot (f \otimes b)$  stand for the  $\hat{\mathfrak{g}}_{-\bar{k}}$ - and  $\hat{\mathfrak{g}}_{\bar{k}}$ -actions on  $S_\gamma \otimes_U \mathbb{V}$  respectively.

To prove the theorem, we need some preparations. First, let

$$\Theta : S \otimes_U \mathbb{V} \rightarrow \mathcal{H}om_B(U, \mathbb{C}_\gamma \otimes \mathbb{B})$$

be the (left)  $U$ -map described in Remark 2.6 (taking  $E' = \mathbb{B}$ ). Note that we regard  $\mathbb{V}, \mathbb{B}$  as non-positively graded, i.e. taking the opposite of the grading defined by the conformal weights of the vertex operator algebra  $\mathbb{V}$ . Here  $\mathbb{B}$  is regarded as a left  $B$ -module via the  $U(\hat{\mathfrak{g}}_{\geq 0}, k)$ -action on  $\mathbb{B}$ , and  $\Theta$  is a  $U(\hat{\mathfrak{g}}, -\bar{k})$ -map.

Following [GMS2, Z1], let  $\tau_i$  be an orthonormal basis of  $\mathfrak{g}$  with respect to the normalized Killing form  $(\cdot, \cdot)$ . Let  $C_{ijk}$  be the structure constants determined by  $[\tau_i, \tau_j] = C_{ijk}\tau_k$ . We identify  $\mathfrak{g}$  with the tangent space to the identity of  $G$ . Let  $\tau_i^L$  (resp.  $\tau_i^R$ ) be the left (resp. right) invariant vector fields valued  $\tau_i$  (resp.  $-\tau_i$ ) at the identity, there exist regular functions  $a^{ij} \in A$  such that  $\tau_i^R = a^{ij}\tau_j^L$  and  $\epsilon(a^{ij}) = -\delta_{ij}$ .

**Lemma 3.2.** *Let  $\beta : B \rightarrow B$  be the automorphism which restricts to  $\hat{\mathfrak{g}}_{\geq 0}$  as  $X \mapsto \gamma(X) + X$ , then for any  $u_{\geq 0} \in B$  and  $b \in \mathbb{B}$ , we have  $\epsilon(\beta(u_{\geq 0})^l \cdot b) = \epsilon(\overline{u_{\geq 0}}^r \cdot b)$ , here  $\beta(u_{\geq 0})^l \cdot b$ ,  $\overline{u_{\geq 0}}^r \cdot b$  denote the  $U(\hat{\mathfrak{g}}_{\geq 0}, k)$ - and  $U(\hat{\mathfrak{g}}_{\geq 0}, \bar{k})$ -actions on  $\mathbb{B}$ .*

*Proof.* By [Z1, Lemma 3.14 (10)], we have  $\tau_j(n)^l \cdot b = \sum_i \sum_{p \geq 0} a_{(-1-p)}^{ij} \tau_i(n+p)^r \cdot b$  for any  $n \geq 0$ ,  $b \in \mathbb{B}$ . Since  $\epsilon|_{B_{\geq 1}} = 0$ , we have  $\epsilon(\tau_j(n)^l \cdot b) = \sum_i \epsilon(a_{(-1)}^{ij}) \tau_i(n)^r \cdot b = \sum_i (-\delta_{ij}) \epsilon(\tau_i(n)^r \cdot b) = -\epsilon(\tau_j(n)^r \cdot b)$ . Since the  $U(\hat{\mathfrak{g}}_{\geq 0}, k)$ - and  $U(\hat{\mathfrak{g}}_{\geq 0}, \bar{k})$ -actions on  $\mathbb{B}$  commute, for any  $u_{\geq 0} = \tau_{j_1}(n_1) \cdots \tau_{j_q}(n_q)$ , we have  $\epsilon(\beta(u_{\geq 0})^l \cdot b) = \epsilon(u_{\geq 0}^l \cdot b) = \epsilon(-\tau_{j_1}(n_1)^r \cdot (\tau_{j_2}(n_2) \cdots)^l \cdot b) = \epsilon((\tau_{j_2}(n_2) \cdots)^l \cdot (-\tau_{j_1}(n_1))^r \cdot b) = \epsilon((\tau_{j_3}(n_3) \cdots)^l \cdot (-\tau_{j_2}(n_2))^r \cdot (-\tau_{j_1}(n_1))^r \cdot b) = \cdots = \epsilon((- \tau_{j_q}(n_q))^r \cdots (-\tau_{j_1}(n_1))^r \cdot b) = \epsilon(\overline{u_{\geq 0}}^r \cdot b)$ . We also have  $\epsilon(\beta(\underline{c})^l \cdot b) = \epsilon((\underline{c} + 2h^\vee)^l \cdot b) = \epsilon((k + 2h^\vee)b) = \epsilon(-\bar{k}b) = \epsilon(\overline{\underline{c}}^r \cdot b)$ , hence the lemma is proved.  $\square$

**Proposition 3.3.** *For any  $f \otimes b \in N^\otimes \otimes_{\mathbb{C}} \mathbb{B}$ , we have  $\Phi(f \otimes b) = \epsilon \Theta(f \otimes b)^-$ .*

*Proof.* By the definition of  $\Theta$ , for any  $u = u_{<0} u_{\geq 0} \in U$ , we have  $\Theta(f \otimes b)(\bar{u}) = \Theta(f \otimes b)(\overline{u_{\geq 0}} \overline{u_{<0}}) = f(\overline{u_{<0}}) \beta(\overline{u_{\geq 0}})^l \cdot b$ . Then it follows from Lemma 3.2 that  $\epsilon \Theta(f \otimes b)(\bar{u}) = f(\overline{u_{<0}}) \epsilon(u_{\geq 0}^r \cdot b) = \Phi(f \otimes b)(u)$ .  $\square$

**Corollary 3.4.** *For any  $u \in U$  and  $f \otimes b \in N^\otimes \otimes_{\mathbb{C}} \mathbb{B}$ , we have  $\Phi(u^l \cdot (f \otimes b)) = (\Phi(f \otimes b)) \cdot \bar{u}$ .*

*Proof.* Since  $\Theta$  is a (left)  $U$ -map, by Proposition 3.3, we have

$$\begin{aligned} \Phi(u^l \cdot (f \otimes b)) &= \epsilon \Theta(u^l \cdot (f \otimes b))^- = \epsilon(u \cdot \Theta(f \otimes b))^- \\ &= \epsilon \Theta(f \otimes b) r_u^- = \epsilon \Theta(f \otimes b)^- l_{\bar{u}} = \Phi(f \otimes b) l_{\bar{u}} = (\Phi(f \otimes b)) \cdot \bar{u}, \end{aligned}$$

where  $r_u, l_{\bar{u}} : U \rightarrow U$  denote the right and left multiplications by  $u$  and  $\bar{u}$  respectively. Hence we proved one half of Theorem 3.1.  $\square$

Next we prove the other half of Theorem 3.1, which is to show that

$$\Phi(u^r \cdot (f \otimes b)) = u \cdot (\Phi(f \otimes b)).$$

If  $u = u_{\geq 0} \in B$ , then  $u_{\geq 0}^r \cdot (f \otimes b) = f \otimes u_{\geq 0}^r \cdot b$ . Hence  $\Phi(f \otimes (u_{\geq 0}^r \cdot b))(u'_{<0} u'_{\geq 0}) = f(\overline{u'_{<0}}) \epsilon(u'_{\geq 0}{}^r \cdot u_{\geq 0}^r \cdot b) = \Phi(f \otimes b)(u'_{<0} u'_{\geq 0} u_{\geq 0}) = u_{\geq 0} \cdot (\Phi(f \otimes b))(u'_{<0} u'_{\geq 0})$ , which means that  $\Phi(u_{\geq 0}^r \cdot (f \otimes b)) = u_{\geq 0} \cdot (\Phi(f \otimes b))$ .

To prove it holds for  $u = u_{<0} \in N$  as well, it suffices to show that  $\Phi(\tau_i(-1)^r \cdot (f \otimes b)) = \tau_i(-1) \cdot (\Phi(f \otimes b))$  since  $\hat{\mathfrak{g}}_{<0}$  is generated by  $\hat{\mathfrak{g}}_{-1}$ .

Recall that although  $\mathbb{B}$  is only closed under the action of  $U(\hat{\mathfrak{g}}_{\geq 0}, \bar{k})$ , it can be equipped with a  $\hat{\mathfrak{g}}_k$ -module structure  $\tilde{\rho} : U \rightarrow \text{End}(\mathbb{B})$  such that  $\tilde{\rho}(u_{\geq 0})b = u_{\geq 0}^r \cdot b$  for any  $u_{\geq 0} \in B$  and  $b \in \mathbb{B}$  (see [Z1, Lemma 3.29, Remark 3.30]). In addition, we have

$$\tau_i(-1)^r \cdot (f \otimes b) = \sum_j f \cdot \tau_j(-1) \otimes (a^{ij}b) + f \otimes \tilde{\rho}(\tau_i(-1))b$$

(see [Z1, Lemma 3.14 (9)]). Hence for any  $u_{<0} \in N$ ,  $u_0 \in U(\hat{\mathfrak{g}}_0)$  and  $u_{>0} \in U(\hat{\mathfrak{g}}_{>0})$ , we have

$$\begin{aligned} & \Phi(\tau_i(-1)^r \cdot (f \otimes b))(u_{<0}u_0u_{>0}) \\ &= \sum_j f(\tau_j(-1)\overline{u_{<0}})\epsilon(u_0^r \cdot u_{>0}^r \cdot (a^{ij}b)) + f(\overline{u_{<0}})\epsilon(u_0^r \cdot u_{>0}^r \cdot \tilde{\rho}(\tau_i(-1))b) \\ &= \sum_j f(\tau_j(-1)\overline{u_{<0}})\epsilon(u_0^r \cdot a_{(-1)}^{ij}u_{>0}^r \cdot b) + f(\overline{u_{<0}})\epsilon(u_0^r \cdot [u_{>0}, \tau_i(-1)]^r \cdot b). \end{aligned}$$

The last equality is because  $[u_{>0}^r, a_{(-1)}^{ij}]|_{\mathbb{B}} = 0$  (see [Z1, Lemma 3.14 (4)]), and  $[u_{>0}, \tau_i(-1)] \in B$ ,  $\epsilon|_{\mathbb{B}_{\geq 1}} = 0$ .

On the other hand, we have

$$\begin{aligned} & \tau_i(-1) \cdot (\Phi(f \otimes b))(u_{<0}u_0u_{>0}) = \Phi(f \otimes b)(u_{<0}u_0u_{>0}\tau_i(-1)) \\ &= \Phi(f \otimes b)(u_{<0}u_0[u_{>0}, \tau_i(-1)] + u_{<0}[u_0, \tau_i(-1)]u_{>0} + u_{<0}\tau_i(-1)u_{\geq 0}) \\ &= f(\overline{u_{<0}})\epsilon(u_0^r \cdot [u_{>0}, \tau_i(-1)]^r \cdot b) + \sum_s f(\overline{u_{<0}\tau_s(-1)})\epsilon(F^{i,s}(u_0)^r \cdot u_{>0}^r \cdot b) \\ & \quad + f(\overline{u_{<0}\tau_i(-1)})\epsilon(u_{\geq 0}^r \cdot b) \end{aligned}$$

where  $F^{i,s} : U(\hat{\mathfrak{g}}_0) \rightarrow U(\hat{\mathfrak{g}}_0)$  are maps such that  $[u_0, \tau_i(-1)] = \sum_s \tau_s(-1)F^{i,s}(u_0)$  for any  $u_0 \in U(\hat{\mathfrak{g}}_0)$ .

Since  $\tau_k^R(a^{ij}) = C_{kip}a^{pj}$ , we have  $[\tau_k(0)^r, a_{(-1)}^{ij}] = C_{kip}a_{(-1)}^{pj}$  (see [Z1, Lemma 3.14 (4)]). Compare it with the commutator  $[\tau_k(0), \tau_i(-1)] = C_{kip}\tau_p(-1)$ , it follows that  $[u_0^r, a_{(-1)}^{ij}] = \sum_s a_{(-1)}^{sj}F^{i,s}(u_0)^r$ . Hence we have

$$\begin{aligned} & \sum_j f(\tau_j(-1)\overline{u_{<0}})\epsilon(u_0^r \cdot a_{(-1)}^{ij}u_{>0}^r \cdot b) \\ &= \sum_j f(\tau_j(-1)\overline{u_{<0}})\epsilon(\sum_s a_{(-1)}^{sj}F^{i,s}(u_0)^r \cdot u_{>0}^r \cdot b) + \sum_j f(\tau_j(-1)\overline{u_{<0}})\epsilon(a_{(-1)}^{ij}u_{\geq 0}^r \cdot b) \\ &= \sum_j f(\tau_j(-1)\overline{u_{<0}})\epsilon(-F^{i,j}(u_0)^r \cdot u_{>0}^r \cdot b) + f(\tau_i(-1)\overline{u_{<0}})\epsilon(-u_{\geq 0}^r \cdot b) \\ &= \sum_j f(\overline{u_{<0}\tau_j(-1)})\epsilon(F^{i,j}(u_0)^r \cdot u_{>0}^r \cdot b) + f(\overline{u_{<0}\tau_i(-1)})\epsilon(u_{\geq 0}^r \cdot b), \end{aligned}$$

which proves that  $\Phi(\tau_i(-1)^r \cdot (f \otimes b)) = \tau_i(-1) \cdot (\Phi(f \otimes b))$ . The proof of Theorem 3.1 is now complete.

**Remark 3.5.** Following the notations in [Z1], let  $\{\tilde{\omega}_i\}$  be right invariant 1-forms dual to  $\{\tau_i^R\}$ , and let  $\widetilde{\mathbb{B}}_0$  be the linear span of elements of the form  $\partial^{(j_1)}\tilde{\omega}_{i_1}\cdots\partial^{(j_n)}\tilde{\omega}_{i_n}$ , then  $\mathbb{B} = A \otimes \widetilde{\mathbb{B}}_0$ . There is a non-degenerate pairing between  $U(\hat{\mathfrak{g}}_{>0})$  and  $\widetilde{\mathbb{B}}_0$ , defined by  $(u_{>0}, \tilde{b}) = \epsilon(u_{>0}^r \cdot \tilde{b})$ , via which  $\widetilde{\mathbb{B}}_0$  can be identified with  $U(\hat{\mathfrak{g}}_{>0})^*$ , the graded dual of  $U(\hat{\mathfrak{g}}_{>0})$ . The regular functions  $A$  can be identified with the Hopf dual  $U(\mathfrak{g})_{\text{Hopf}}^*$ , which is a subalgebra of  $U(\mathfrak{g})^*$  defined by

$$U(\mathfrak{g})_{\text{Hopf}}^* = \{\phi \in U(\mathfrak{g})^* \mid \text{Ker}\phi \text{ contains a two-sided ideal } J \subset U(\mathfrak{g}) \text{ of finite codimension}\}.$$

It is not hard to see that  $\epsilon(u_0^r \cdot u_{>0}^r \cdot a\tilde{b}) = \epsilon(u_0^r \cdot a)\epsilon(u_{>0}^r \cdot \tilde{b})$  for any  $u_0 \in U(\mathfrak{g})$ ,  $u_{>0} \in U(\hat{\mathfrak{g}}_{>0})$ ,  $a \in A$  and  $\tilde{b} \in \widetilde{\mathbb{B}}_0$ . Hence

$$S \otimes_U \mathbb{V} \cong N^* \otimes \mathbb{B} \cong N^* \otimes A \otimes \widetilde{\mathbb{B}}_0 \cong U(\hat{\mathfrak{g}}_{<0})^* \otimes U(\mathfrak{g})_{\text{Hopf}}^* \otimes U(\hat{\mathfrak{g}}_{>0})^* \subset U(\hat{\mathfrak{g}}, \bar{k})^*,$$

and  $\Phi$  is injective.

#### 4. FILTRATIONS OF THE VERTEX OPERATOR ALGEBRA $\mathbb{V}$

Fix  $k \in \mathbb{Q}$ ,  $k > -h^\vee$ ; set  $\varkappa = k + h^\vee > 0$ . Let  $\mathcal{O}_{-\varkappa}$  be the full subcategory of the category of  $\hat{\mathfrak{g}}_{\bar{k}}$ -modules defined by Kazhdan and Lusztig in [KL1-4]. They constructed a tensor structure on  $\mathcal{O}_{-\varkappa}$ , and established an equivalence of tensor categories between  $\mathcal{O}_{-\varkappa}$  and the category of finite-dimensional integrable representations of the quantum group with quantum parameter  $q = e^{-i\pi/\varkappa}$  (in the simply-laced case).

Let  $V_{\lambda, \bar{k}} = \text{Ind}_{\hat{\mathfrak{g}}_{\geq 0}}^{\hat{\mathfrak{g}}} V_\lambda$  be a Weyl module, denote the irreducible quotient of  $V_{\lambda, \bar{k}}$  by  $L_{\lambda, \bar{k}}$ .

**Definition 4.1.** [KL1, Definition 2.15]  $\mathcal{O}_{-\varkappa}$  is the full subcategory of  $\hat{\mathfrak{g}}_{\bar{k}}$ -modules, which admits a finite composition series with factors of the form  $L_{\lambda, \bar{k}}$  for various  $\lambda \in P^+$ .

Let us recall some basic facts about  $\mathcal{O}_{-\varkappa}$ . The  $\mathbb{Z}_{>0}$ -grading on  $\hat{\mathfrak{g}}_{>0}$  induces an  $\mathbb{N}$ -grading on the enveloping algebra:  $U(\hat{\mathfrak{g}}_{>0}) = \bigoplus_{n \geq 0} U(\hat{\mathfrak{g}}_{>0})_n$ . For any  $V \in \mathcal{O}_{-\varkappa}$ ,  $v \in V$ , there exists an  $n_1 \in \mathbb{N}$  such that  $U(\hat{\mathfrak{g}}_{>0})_{n_1} \cdot v = 0$ .

A module  $\mathcal{N}$  over  $\mathfrak{g} \otimes \mathbb{C}[t]$  is said to be a nil-module if  $\dim_{\mathbb{C}} \mathcal{N} < \infty$  and there exists a  $n \geq 1$  such that  $U(\hat{\mathfrak{g}}_{>0})_n \mathcal{N} = 0$ . Extend  $\mathcal{N}$  to a  $\hat{\mathfrak{g}}_{\geq 0}$ -module by defining the action of  $\underline{c}$  to be multiplication by  $\bar{k}$ , and let  $\mathcal{N}_{\bar{k}} = \text{Ind}_{\hat{\mathfrak{g}}_{\geq 0}}^{\hat{\mathfrak{g}}} \mathcal{N}$  be the induced module. We say that  $\mathcal{N}_{\bar{k}}$  is a generalized Weyl module.

**Proposition 4.2.** [KL1, Theorem 2.22] *A  $\hat{\mathfrak{g}}_{\bar{k}}$ -module  $V$  is in  $\mathcal{O}_{-\varkappa}$  if and only if  $V$  is a quotient of a generalized Weyl module.*

Given  $V \in \mathcal{O}_{-\varkappa}$ , let  $\bar{L}_0 : V \rightarrow V$  be the Sugawara operator defined by  $\bar{L}_0 v = -\frac{1}{\varkappa} \sum_{j>0} \sum_i \tau_i(-j) \tau_i(j) v - \frac{1}{2\varkappa} \sum_i \tau_i(0) \tau_i(0) v$ , where  $\{\tau_i\}$  is an orthonormal basis of  $\mathfrak{g}$  with respect to the normalized Killing form. Note that this operator is well defined and locally finite. Let  $V_z$  be the generalized eigenspace of  $\bar{L}_0$  with eigenvalue  $-z \in \mathbb{C}$ , we have  $V = \bigoplus_{z \in \mathbb{C}} V_z$  with  $\dim V_z < \infty$ . In fact there exist  $z_1, \dots, z_m \in \mathbb{Q}$  such that  $\{z \mid V_z \neq 0\} \subset \{z_1 - \mathbb{N}\} \cup \dots \cup \{z_m - \mathbb{N}\}$ , and  $V$  becomes a  $\mathbb{Q}$ -graded  $\hat{\mathfrak{g}}_{\bar{k}}$ -representation, i.e.  $x(n)V_z \subset V_{z+n}$  for any  $x(n) \in \hat{\mathfrak{g}}$  (see [KL1, Lemma 2.20, Proposition 2.21]). In



case  $V = V_{\lambda, \bar{k}}$  is a Weyl module,  $\bar{L}_0$  acts on  $V_{\lambda, \bar{k}}$  semisimply. More specifically, we have  $\bar{L}_0|_{U(\hat{\mathfrak{g}}_{<0})-n \otimes V_\lambda} = -\frac{\langle \lambda, \lambda + 2\rho \rangle}{2\kappa} + n$ , where  $\rho$  is the half sum of positive roots.

Define the dual representation of  $V$  as follows: as a vector space  $V^* = \bigoplus_z (V_z)^*$ ; the  $\hat{\mathfrak{g}}$ -action is given by  $Xf(v) = f(-Xv)$  for any  $X \in \hat{\mathfrak{g}}, f \in V^*, v \in V$ . In particular  $V^*$  is a  $\hat{\mathfrak{g}}_{-\bar{k}}$ -module and locally  $U(\hat{\mathfrak{g}}_{<0})$ -finite. In order for  $V^*$  to be a graded  $\hat{\mathfrak{g}}$ -module as well, set  $(V^*)_z = (V_{-z})^*$ , or equivalently set  $(V^*)_z$  to be the generalized  $(-z)$ -eigenspace of the operator  $L'_0 = \frac{1}{\kappa} \sum_{j>0} \sum_i \tau_i(j) \tau_i(-j) + \frac{1}{2\kappa} \sum_i \tau_i(0) \tau_i(0)$  which acts on  $V^*$ .

The contragredient dual  $V^c$  is isomorphic to  $V^*$  as a vector space, but instead of using  $-\text{Id} : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ , we use the anti-involution  $x(n) \mapsto -x(-n), \underline{c} \mapsto \underline{c}$  to define the  $\hat{\mathfrak{g}}$ -action on  $V^c$ . Unlike  $V^*$ , the contragredient module  $V^c$  is a  $\hat{\mathfrak{g}}_{\bar{k}}$ -representation, locally  $U(\hat{\mathfrak{g}}_{>0})$ -finite, and in fact belongs to  $\mathcal{O}_{-\kappa}$ .

Given  $V \in \mathcal{O}_{-\kappa}$ , define a map  $\phi_V : V^* \otimes V \rightarrow U(\hat{\mathfrak{g}}, \bar{k})^*$ ;  $\phi_V(f \otimes v)(u) = f(u \cdot v)$  for any  $f \in V^*, v \in V, u \in U(\hat{\mathfrak{g}})$ . It is easy to see that  $\phi_V$  is a  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -map, where the  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -module structure of  $U(\hat{\mathfrak{g}}, \bar{k})^*$  is given by  $(X, 0) \cdot g = -g \cdot X$  and  $(0, X) \cdot g = X \cdot g$  for any  $X \in \hat{\mathfrak{g}}, g \in U(\hat{\mathfrak{g}}, \bar{k})^*$ . Denote the image of  $\phi_V$  by  $\mathbb{M}(V)$ , which is called the matrix coefficients of  $V$ .

Recall the  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -map  $\Phi : S_\gamma \otimes_U \mathbb{V} \rightarrow U(\hat{\mathfrak{g}}, \bar{k})^*$  defined in Section 3. As pointed out in Remark 3.5, the map  $\Phi$  is injective and its image, which we denote by  $\mathbb{M}^{\mathcal{O}_{-\kappa}}$ , is isomorphic to  $U(\hat{\mathfrak{g}}_{<0})^{\otimes} \otimes U(\mathfrak{g})_{\text{Hopf}}^* \otimes U(\hat{\mathfrak{g}}_{>0})^{\otimes}$ . Here  $U(\hat{\mathfrak{g}}_{<0})^{\otimes} = \bigoplus_{n \leq 0} (U(\hat{\mathfrak{g}}_{<0})_n)^*$ ,  $U(\hat{\mathfrak{g}}_{>0})^{\otimes} = \bigoplus_{n \geq 0} (U(\hat{\mathfrak{g}}_{>0})_n)^*$  are graded duals.

**Proposition 4.3.**  $\mathbb{M}^{\mathcal{O}_{-\kappa}}$  consists of matrix coefficients of modules from the category  $\mathcal{O}_{-\kappa}$ , i.e.  $\mathbb{M}^{\mathcal{O}_{-\kappa}} = \sum_{V \in \mathcal{O}_{-\kappa}} \mathbb{M}(V)$ .

*Proof.* Let  $V = \bigoplus_z V_z \in \mathcal{O}_{-\kappa}$ ,  $v \in V$  and  $f \in V^*$ , for any  $u = u_{<0} u_0 u_{>0} \in U = U(\hat{\mathfrak{g}})$ , we have  $\phi_V(f \otimes v)(u) = \langle f, u_{<0} u_0 u_{>0} \cdot v \rangle = \langle \overline{u_{<0}} \cdot f, u_0 \cdot u_{>0} \cdot v \rangle$ . Since  $V \in \mathcal{O}_{-\kappa}$ , there exist  $n_1, n_2 \in \mathbb{N}$  such that  $U(\hat{\mathfrak{g}}_{<0})_{-n_1} \cdot f = U(\hat{\mathfrak{g}}_{>0})_{n_2} \cdot v = 0$ . Moreover each  $V_z$  is finite-dimensional and semisimple as a  $\mathfrak{g}$ -module, therefore it is not hard to see that  $\phi_V(f \otimes v) \in U(\hat{\mathfrak{g}}_{<0})^{\otimes} \otimes U(\mathfrak{g})_{\text{Hopf}}^* \otimes U(\hat{\mathfrak{g}}_{>0})^{\otimes}$ , i.e.  $\mathbb{M}(V) \subset \mathbb{M}^{\mathcal{O}_{-\kappa}}$ .

On the other hand, let  $g \in \mathbb{M}^{\mathcal{O}_{-\kappa}}$ , there exists an  $n \in \mathbb{N}$  such that  $U(\hat{\mathfrak{g}}_{>0})_n \cdot g = 0$ . Since each  $U(\hat{\mathfrak{g}}_{>0})_{n'}$  is finite-dimensional and  $\mathfrak{g}$  acts on  $\mathbb{M}^{\mathcal{O}_{-\kappa}}$  locally finitely, the  $\hat{\mathfrak{g}}_{\geq 0}$ -submodule generated by  $g$  is a nil-module. Hence the  $\hat{\mathfrak{g}}$ -submodule  $W = U(\hat{\mathfrak{g}}) \cdot g$  generated by  $g$  is a quotient of a generalized Weyl module, hence it belongs to  $\mathcal{O}_{-\kappa}$ . Let  $\delta$  be the functional on  $U^*$  defined by  $\delta(g') = g'(1)$ , then  $\delta \in W^*$  and  $g = \phi_W(\delta \otimes g) \in \mathbb{M}(W)$ .  $\square$

Define two operators  $\bar{L}_0, L'_0$  that act on  $\mathbb{M}^{\mathcal{O}_{-\kappa}}$  as follows: for any  $g \in \mathbb{M}^{\mathcal{O}_{-\kappa}}$ , set  $\bar{L}_0 g = -\frac{1}{\kappa} \sum_{j>0} \sum_i \tau_i(-j) \cdot \tau_i(j) \cdot g - \frac{1}{2\kappa} \sum_i \tau_i(0) \cdot \tau_i(0) \cdot g$  and  $L'_0 g = \frac{1}{\kappa} \sum_{j>0} \sum_i g \cdot \tau_i(-j) \cdot \tau_i(j) + \frac{1}{2\kappa} \sum_i g \cdot \tau_i(0) \cdot \tau_i(0)$ . Let  $\mathbb{M}_{z', z}^{\mathcal{O}_{-\kappa}}$  be the subspace consisting of all  $g \in \mathbb{M}^{\mathcal{O}_{-\kappa}}$  such that  $g$  is in the kernel of some power of  $\bar{L}_0 + z \text{Id}$  and the kernel of some power of  $L'_0 + z' \text{Id}$ . Then  $\mathbb{M}^{\mathcal{O}_{-\kappa}} = \bigoplus_{z, z'} \mathbb{M}_{z', z}^{\mathcal{O}_{-\kappa}}$ , and  $\phi_V((V^*)_{z'} \otimes V_z) \subset \mathbb{M}_{z', z}^{\mathcal{O}_{-\kappa}}$  for any  $V \in \mathcal{O}_{-\kappa}$ . Moreover  $\mathbb{M}_{z', z}^{\mathcal{O}_{-\kappa}} \cdot x(n) \subset \mathbb{M}_{z'+n, z}^{\mathcal{O}_{-\kappa}}$  and  $x(n) \cdot \mathbb{M}_{z', z}^{\mathcal{O}_{-\kappa}} \subset \mathbb{M}_{z', z+n}^{\mathcal{O}_{-\kappa}}$  for any  $x(n) \in \hat{\mathfrak{g}}$ . Define a  $\mathbb{Z}$ -grading on  $\mathbb{M}^{\mathcal{O}_{-\kappa}}$ : for any  $g_1 \in (U(\hat{\mathfrak{g}}_{<0})_n)^*$ ,  $a \in U(\mathfrak{g})_{\text{Hopf}}^*$ ,  $g_2 \in (U(\hat{\mathfrak{g}}_{>0})_{n'})^*$ , define  $\deg g_1 \otimes a \otimes g_2 = -n - n'$ ; set  $\mathbb{M}_n^{\mathcal{O}_{-\kappa}} = \{g \mid \deg g = n\}$ . It is not difficult to see that  $\mathbb{M}_n^{\mathcal{O}_{-\kappa}} = \bigoplus_{z+z'=n} \mathbb{M}_{z', z}^{\mathcal{O}_{-\kappa}}$ .

Following [KL1, 3.3], define a partial order on  $P^+$  as follows:  $\lambda \leq \mu$  if either  $\lambda = \mu$  or  $\langle \lambda, \lambda + 2\rho \rangle < \langle \mu, \mu + 2\rho \rangle$ . Let  $\mathcal{O}_{-\varkappa}^s$  be the full subcategory of  $\mathcal{O}_{-\varkappa}$  whose objects are the  $V$  in  $\mathcal{O}_{-\varkappa}$  such that the composition factors of  $V$  are of the form  $L_{\lambda, \bar{k}}$  for some  $\lambda$  in the finite set  $F^s = \{\lambda \in P^+ | \langle \lambda, \lambda + 2\rho \rangle \leq s\}$ .

We say that a module  $V \in \mathcal{O}_{-\varkappa}$  is tilting if both  $V$  and  $V^c$  have a Weyl filtration. For any  $\lambda \in P^+$ , there exists an indecomposable tilting module  $T_{\lambda, \bar{k}}$  such that  $V_{\lambda, \bar{k}} \hookrightarrow T_{\lambda, \bar{k}}$ , and any other Weyl modules  $V_{\mu, \bar{k}}$  entering the Weyl filtration of  $T_{\lambda, \bar{k}}$  satisfy  $\mu < \lambda$  (see [KL4, Proposition 27.2]).

**Lemma 4.4.** *Let  $V, V' \in \mathcal{O}_{-\varkappa}$ .*

- (1) *If  $V$  has a (finite) Weyl filtration with factors isomorphic to  $V_{\lambda_i, \bar{k}}$  for various  $\lambda_i \in P^+$ , then  $\mathbb{M}(V) \subset \sum_i \mathbb{M}(T_{\lambda_i, \bar{k}})$ .*
- (2) *If  $V'$  has a (finite) filtration with factors isomorphic to  $V_{\mu_i, \bar{k}}^c$  for various  $\mu_i \in P^+$ , then  $\mathbb{M}(V') \subset \sum_i \mathbb{M}(T_{\mu_i, \bar{k}}^c)$ .*

*Proof.* The proof is exactly the same as that of [Z2, Lemma 3.2]: we can construct an injection  $V \hookrightarrow \bigoplus_i T_{\lambda_i, \bar{k}}$ , and a surjection  $\bigoplus_i T_{\mu_i, \bar{k}}^c \twoheadrightarrow V'$ , since  $\text{Ext}_{\mathcal{O}_{-\varkappa}}^1(V_{\lambda, \bar{k}}, V_{\mu, \bar{k}}^c) = 0$  (see [KL4, Proposition 27.1]).  $\square$

**Corollary 4.5.**  *$\mathbb{M}^{\mathcal{O}_{-\varkappa}}$  consists of the matrix coefficients of tilting modules from  $\mathcal{O}_{-\varkappa}$ , i.e.  $\mathbb{M}^{\mathcal{O}_{-\varkappa}} = \sum_{V \in \mathcal{O}_{-\varkappa}, V \text{ tilting}} \mathbb{M}(V)$ .*

*Proof.* For any  $V \in \mathcal{O}_{-\varkappa}$ , choose  $s$  such that  $V \in \mathcal{O}_{-\varkappa}^s$ . By [KL1, Proposition 3.9], there exists a  $P$ , projective in  $\mathcal{O}_{-\varkappa}^s$  and having a (finite) Weyl filtration, such that  $V$  is a quotient of  $P$ . Hence by Lemma 4.4 (1), we have  $\mathbb{M}(V) \subset \mathbb{M}(P) \subset \sum_i \mathbb{M}(T_{\lambda_i, \bar{k}})$  for some  $\lambda_i \in F^s$ .  $\square$

**Proposition 4.6.** *Order the dominant weights in such a way  $P^+ = \{\nu_1, \dots, \nu_i, \dots\}$  that  $\nu_i < \nu_j$  implies  $i < j$ . Set  $\mathbb{M}^{\mathcal{O}_{-\varkappa}, i} = \sum_{j \leq i} \mathbb{M}(T_{\nu_j, \bar{k}})$ , then  $\mathbb{M}^{\mathcal{O}_{-\varkappa}, 1} \subset \dots \subset \mathbb{M}^{\mathcal{O}_{-\varkappa}, i-1} \subset \mathbb{M}^{\mathcal{O}_{-\varkappa}, i} \subset \dots$  is an increasing filtration of  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -submodules of  $\mathbb{M}^{\mathcal{O}_{-\varkappa}}$  with factors  $\mathbb{M}^{\mathcal{O}_{-\varkappa}, i} / \mathbb{M}^{\mathcal{O}_{-\varkappa}, i-1}$  isomorphic to  $V_{\nu_i, \bar{k}}^* \otimes V_{-\omega_0 \nu_i, \bar{k}}^c$ , where  $\omega_0$  is the longest element in the Weyl group.*

*Proof.* The proof is the same as that of [Z2, Theorem 3.3], using Lemma 4.4.  $\square$

**Remark 4.7.** The category  $\mathcal{O}_{-\varkappa}$  is a direct sum of subcategories corresponding to the orbits of the shifted action of affine Weyl group on the weight lattice (see [KL4, Lemma 27.7]). Hence we can decompose  $\mathbb{M}^{\mathcal{O}_{-\varkappa}}$ , as a  $\hat{\mathfrak{g}}_{-\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -module, into summands corresponding to the orbits as well. Some summands are semisimple (see [KL4, Proposition 27.4], [Z2, Proposition 3.1]), but all have an increasing filtration of the above type.

**Proposition 4.8.** *The vertex operator algebra  $\mathbb{V}$  is isomorphic to  $\mathcal{H}om_U(S_\gamma, \mathbb{M}^{\mathcal{O}_{-\varkappa}})$  as a  $\hat{\mathfrak{g}}_{\bar{k}} \oplus \hat{\mathfrak{g}}_{-\bar{k}}$ -module.*

*Proof.* Recall that  $\mathbb{M}^{\mathcal{O}_{-\varkappa}} \cong S_\gamma \otimes_U \mathbb{V} = N^* \otimes \mathbb{B}$ . Hence  $\mathcal{H}om_U(S_\gamma, \mathbb{M}^{\mathcal{O}_{-\varkappa}}) \cong \mathcal{H}om_N(N^*, N^* \otimes \mathbb{B}) \cong \mathcal{H}om_{\mathbb{C}}(N^*, \mathbb{B}) \cong N \otimes \mathbb{B} \cong \mathbb{V}$ , the second to last isomorphism is because  $\mathbb{B}$  is non-positively graded while  $N^*$  is non-negatively graded. Moreover the induced isomorphism  $\mathbb{V} \rightarrow \mathcal{H}om_U(S_\gamma, \mathbb{M}^{\mathcal{O}_{-\varkappa}}) \cong \mathcal{H}om_U(S_\gamma, S_\gamma \otimes_U \mathbb{V})$  is a  $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ -map.  $\square$

**Lemma 4.9.** *For any  $b \in \mathbb{B}$ , there exists an  $i$  such that  $N^* \otimes b \subset \mathbb{M}^{\mathcal{O}_{-\varkappa}, i}$ .*

*Proof.* For any  $f \in N^\otimes$  and  $u_{\geq 0} \in U(\hat{\mathfrak{g}}_{\geq 0})$ , we have  $u_{\geq 0} \cdot (f \otimes b) = f \otimes (u_{\geq 0}^r \cdot b)$ . Let  $\mathcal{N}$  be the  $U(\hat{\mathfrak{g}}_{\geq 0}, \bar{k})$ -submodule of  $\mathbb{B}$  generated by  $b$ , then  $\mathcal{N}$  is a nil-module and the  $\hat{\mathfrak{g}}_{\bar{k}}$ -submodule  $U(\hat{\mathfrak{g}}, \bar{k}) \cdot (f \otimes b)$  generated by  $f \otimes b$  is a quotient of the generalized Weyl module  $\mathcal{N}_{\bar{k}}$ . Hence  $f \otimes b \in \mathbb{M}(\mathcal{N}_{\bar{k}})$  for any  $f \in N^\otimes$ , hence there exists an  $i$  such that  $N^\otimes \otimes b \subset \mathbb{M}^{O_{-\infty}, i}$ .  $\square$

**Theorem 4.10.** *Set  $\Sigma^i = \text{Hom}_U(S_\gamma, \mathbb{M}^{O_{-\infty}, i})$ , then  $\mathbb{V} = \bigcup_i \Sigma^i$  and  $\Sigma^1 \subset \cdots \subset \Sigma^{i-1} \subset \Sigma^i \subset \cdots$  is an increasing filtration of  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -submodules of  $\mathbb{V}$  with factors  $\Sigma^i / \Sigma^{i-1}$  isomorphic to  $V_{-\omega_0 \nu_i, k} \otimes V_{-\omega_0 \nu_i, \bar{k}}^c$ .*

*Proof.* For any  $u_{< 0} \otimes b \in N \otimes \mathbb{B} \cong \mathbb{V}$ , let  $\mathcal{N}' \subset \mathbb{B}$  be the  $U(\hat{\mathfrak{g}}_{\geq 0}, k)$ -submodule generated by  $b$ , then  $\mathcal{N}'$  is finite-dimensional. For any  $s \in S_\gamma$  we have  $p(s \otimes (u_{< 0} \otimes b)) \in N^\otimes \otimes \mathcal{N}'$ , where  $p : S_\gamma \otimes \mathbb{V} \rightarrow S_\gamma \otimes_U \mathbb{V}$  is the canonical projection. By Lemma 4.9, there exists an  $i$  such that  $p(s \otimes (u_{< 0} \otimes b)) \in \mathbb{M}^{O_{-\infty}, i}$  for any  $s \in S_\gamma$ , hence  $u_{< 0} \otimes b \in \text{Hom}_U(S_\gamma, \mathbb{M}^{O_{-\infty}, i}) = \Sigma^i$ . This proves that  $\mathbb{V} = \bigcup_i \Sigma^i$ .

Note that  $\mathbb{M}^{O_{-\infty}, i} = \bigoplus_{z', z} \mathbb{M}_{z', z}^{O_{-\infty}, i}$  with  $\dim \mathbb{M}_{z', z}^{O_{-\infty}, i} < \infty$ . Fix  $z$ , the exact sequence of  $\hat{\mathfrak{g}}_{\bar{k}} \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -modules  $0 \rightarrow \mathbb{M}^{O_{-\infty}, i-1} \rightarrow \mathbb{M}^{O_{-\infty}, i} \rightarrow V_{\nu_i, \bar{k}}^* \otimes V_{-\omega_0 \nu_i, \bar{k}}^c \rightarrow 0$  restricts to an exact sequence of  $\hat{\mathfrak{g}}_{\bar{k}}$ -modules  $0 \rightarrow \bigoplus_{z'} \mathbb{M}_{z', z}^{O_{-\infty}, i-1} \rightarrow \bigoplus_{z'} \mathbb{M}_{z', z}^{O_{-\infty}, i} \rightarrow V_{\nu_i, \bar{k}}^* \otimes (V_{-\omega_0 \nu_i, \bar{k}}^c)_z \rightarrow 0$ . Since  $V_{\nu_i, \bar{k}}^* \otimes (V_{-\omega_0 \nu_i, \bar{k}}^c)_z$  is isomorphic to a finite direct sum of grading-shifted copies of  $N^\otimes$  over  $N$ , by induction on  $i$ , so does  $\bigoplus_{z'} \mathbb{M}_{z', z}^{O_{-\infty}, i}$  for each  $i$ , and the two exact sequences split over  $N$ , which means that there exists a grading preserving  $N$ -map  $V_{\nu_i, \bar{k}}^* \otimes V_{-\omega_0 \nu_i, \bar{k}}^c \rightarrow \mathbb{M}^{O_{-\infty}, i}$  so that its composition with the projection is identity on the former. Therefore the sequence of  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -modules  $0 \rightarrow \text{Hom}_U(S_\gamma, \mathbb{M}^{O_{-\infty}, i-1}) \rightarrow \text{Hom}_U(S_\gamma, \mathbb{M}^{O_{-\infty}, i}) \rightarrow \text{Hom}_U(S_\gamma, V_{\nu_i, \bar{k}}^* \otimes V_{-\omega_0 \nu_i, \bar{k}}^c) \rightarrow 0$  is exact since  $\text{Hom}_U(S_\gamma, -) \cong \text{Hom}_N(N^\otimes, -)$ . Hence we have  $\Sigma^i / \Sigma^{i-1} \cong \text{Hom}_U(S_\gamma, V_{\nu_i, \bar{k}}^* \otimes V_{-\omega_0 \nu_i, \bar{k}}^c)$ , which is isomorphic to  $V_{-\omega_0 \nu_i, k} \otimes V_{-\omega_0 \nu_i, \bar{k}}^c$  by Proposition 2.7, Lemma 2.8 and the fact that the grading on  $V_{-\omega_0 \nu_i, \bar{k}}^c$  is bounded from above.  $\square$

**Remark 4.11.** The decomposition of  $\mathbb{M}^{O_{-\infty}}$  discussed in Remark 4.7 leads to a decomposition of  $\mathbb{V}$ , as a  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -module, into summands corresponding to the orbits of the affine Weyl group on the weight lattice. Again some summands are semisimple, but each has an increasing filtration of the above type.

**Corollary 4.12.** *The vertex operator algebra  $\mathbb{V}$  admits a decreasing filtration of  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -submodules  $\mathbb{V} \supset \Xi_1 \supset \cdots \supset \Xi_{i-1} \supset \Xi_i \supset \cdots$  with factors  $\Xi_{i-1} / \Xi_i$  isomorphic to  $V_{-\omega_0 \nu_i, k} \otimes V_{-\omega_0 \nu_i, \bar{k}}^c$ , and  $\bigcap_i \Xi_i = 0$ .*

*Proof.* Let  $L_0, \bar{L}_0 : \mathbb{V} \rightarrow \mathbb{V}$  be the Sugawara operators associated to the  $\hat{\mathfrak{g}}_k$ - and  $\hat{\mathfrak{g}}_{\bar{k}}$ -actions on  $\mathbb{V}$  respectively, i.e.  $L_0 = \frac{1}{2\kappa} \sum_{j>0} \sum_i \tau_i(-j) \tau_i(j) + \frac{1}{2\kappa} \sum_i \tau_i(0) \tau_i(0)$ , and  $\bar{L}_0 = -\frac{1}{2\kappa} \sum_{j>0} \sum_i \bar{\tau}_i(-j) \bar{\tau}_i(j) - \frac{1}{2\kappa} \sum_i \bar{\tau}_i(0) \bar{\tau}_i(0)$ . Now we regard the vertex operator algebra  $\mathbb{V} = \bigoplus_{n \geq 0} \mathbb{V}_n$  as non-negatively graded, then the sum  $\mathcal{L}_0 = L_0 + \bar{L}_0$  is the gradation operator, i.e.  $\mathcal{L}_0|_{\mathbb{V}_n} = n \text{Id}$  (see [Z1, Proposition 3.20, 3.24]).

Let  $\mathbb{V}_{z_1, z_2}$  be the subspace consisting of  $v \in \mathbb{V}$  such that  $v$  is killed by some power of  $L_0 - z_1 \text{Id}$  and some power of  $\bar{L}_0 - z_2 \text{Id}$ . It follows from Theorem 4.10 that  $\mathbb{V} = \bigoplus_{z_1, z_2} \mathbb{V}_{z_1, z_2}$  with  $\dim \mathbb{V}_{z_1, z_2} < \infty$ .

Recall the symmetric non-degenerate bilinear form  $\langle, \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$  constructed in [Z1, Proposition 3.28]. It is shown to be compatible with the vertex operator

algebra structure of  $\mathbb{V}$ , in particular we have  $\langle x(n)\cdot, \cdot \rangle = \langle \cdot, -x(-n)\cdot \rangle$  and  $\langle \bar{y}(n)\cdot, \cdot \rangle = \langle \cdot, -\bar{y}(-n)\cdot \rangle$  for any  $x(n) \in \hat{\mathfrak{g}}_k, \bar{y}(n) \in \hat{\mathfrak{g}}_{\bar{k}}$ . It implies that  $\langle L_0\cdot, \cdot \rangle = \langle \cdot, L_0\cdot \rangle$  and  $\langle \bar{L}_0\cdot, \cdot \rangle = \langle \cdot, \bar{L}_0\cdot \rangle$ . Hence  $\langle \cdot, \cdot \rangle|_{\mathbb{V}_{z_1, z_2} \times \mathbb{V}_{z'_1, z'_2}} = 0$  except when  $z_1 = z'_1$  and  $z_2 = z'_2$ , in which case the pairing is non-degenerate.

Let  $\mathbb{V}^c = \bigoplus_{z_1, z_2} \mathbb{V}_{z_1, z_2}^*$  be the contragredient dual of  $\mathbb{V}$ , where the  $\hat{\mathfrak{g}}_k$ - and  $\hat{\mathfrak{g}}_{\bar{k}}$ -actions on  $\mathbb{V}^c$  are both defined by the anti-involution  $x(n) \mapsto -x(-n); \underline{c} \mapsto \underline{c}$  of  $\hat{\mathfrak{g}}$ . Then we have  $\mathbb{V} \cong \mathbb{V}^c$  because of the bilinear form  $\langle \cdot, \cdot \rangle$ .

Set  $\Xi_i = \{v \in \mathbb{V} \mid \langle v, \Sigma_i \rangle = 0\}$ , then  $\Xi_i$  is a  $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$ -submodule of  $\mathbb{V}$ . Moreover we have  $\Xi_i \subset \Xi_{i-1}$ , and  $\bigcap_i \Xi_i = 0$  because  $\bigcup_i \Sigma_i = \mathbb{V}$  and  $\langle \cdot, \cdot \rangle$  is non-degenerate. In fact  $\Xi_{i-1}/\Xi_i \cong (\Sigma_i/\Sigma_{i-1})^c \cong (V_{-\omega_0\nu_i, k} \otimes V_{-\omega_0\nu_i, \bar{k}}^c)^c \cong V_{-\omega_0\nu_i, k}^c \otimes V_{-\omega_0\nu_i, \bar{k}}$ .  $\square$

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